THE UNIVERSITY OF ALBERTA

CLASSIFICATION OF THE TRACEFREE RICCI TENSOR

BY

© GERARD PATRICK SCANLAN

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled CLASSIFICATION OF THE TRACEFREE RICCI TENSOR submitted by Gerard Patrick Scanlan in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

The aim of this dissertation is to classify the tracefree Ricci tensor. In Chapter I, a brief resume of spinor algebra and spinor analysis is given. Chapters II and III contain the classification of the Weyl tensor and of a real vector, respectively. The tracefree Ricci tensor is classified in Chapter III by a spinor method. The final chapter contains an application of this classification. Necessary and sufficient conditions are found for the geometry of space-time to have a real scalar field as its source.

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INTRODUCTION

In this thesis, the tracefree Ricci tensor will be classified by a method similar to the spinor method used in the Petrov classification of a Weyl tensor. The Petrov classification has found wide applications in connection with the propagation of gravitational fields. We shall give an example to show how the analogous classification of the tracefree Ricci tensor can be used in geometrodynamics.

An introduction to spinor algebra and spinor analysis is given in Chapter I. No attempt is made to prove the results of this chapter, but frequently we shall indicate how the results may be derived.

In Chapter II, we review the Petrov classification of the Weyl tensor as a means of introducing the spinor method of classification. In Chapter III, a real vector is classified by this method.

The main result of this dissertation is achieved in Chapter IV with the classification of the tracefree Ricci tensor. Four distinct classes are established, with each of the first three discussed in some detail.

As an application of this classification, we find necessary and sufficient conditions for the geometry of

space-time to have a real scalar field as its source. These conditions are derived using spinor techniques. Similar conditions were derived by R. Penney and D. R. Brill. However, they assumed that the Ricci scalar, R, was positive; in addition, the differential conditions they obtained vanish identically for R=0. The conditions on the geometry obtained in this thesis are valid for all values of R.

CHAPTER I

A BRIEF REVIEW OF SPINOR ALGEBRA AND SPINOR ANALYSIS

1. Introduction:

In this chapter we shall give a brief introduction to spinor algebra and spinor analysis. In Section 2, some of the notation used in this and the following chapters is explained. Section 3 contains the definition of spinors and their corresponding transformation laws. Some spinor algebra is discussed in Section 4. The definition of a dyad, dyad components, and dyad transformations is also given. The correspondence between tensors and spinors is given in Section 5. Some important spinor equivalents are also listed. In the final section, we briefly discuss the covariant derivative in spinor form, and define spin coefficients. Some important spinor differential relations are given.

2. Notation:

Let us briefly explain the notation used in this and the following chapters. Small Greek letters will be used for tensor indices, and will run from 1 to 4. Capital and small Latin letters will be used to denote spinor indices and dyad indices, respectively; they will run from 1 to 2. The summation convention applies throughout. A bar over a quantity represents the complex conjugate of that quantity. Round brackets enclosing

suffixes will be used for symmetrization, and square brackets for antisymmetrization.

The Levi-Civita alternation symbol, in two dimensions, is designated by ε_{AB}. It is specified by the equation,

$$\varepsilon_{AB} = \varepsilon_{[AB]}, \text{ with } \varepsilon_{12} = 1.$$

The signature of space-time is taken to be -2. The sign of the Riemann tensor is specified by the Ricci identity

$$\nabla_{[\alpha}\nabla_{\beta]}T_{\gamma} = 1/2\ R^{\delta}{}_{\gamma\alpha\beta}T_{\delta},$$

where ∇_{α} denotes the covariant derivative. The Ricci tensor is defined by

$$R_{\alpha\beta} = R^{\delta}{}_{\alpha\delta\beta},$$

and the Ricci scalar by

$$R = R^{\alpha}{}_{\alpha}.$$

Einstein's field equations are taken to be

$$R_{\alpha\beta} - 1/2\ Rg_{\alpha\beta} = -T_{\alpha\beta}, \tag{1.1}$$

where $T_{\alpha\beta}$ is the energy-momentum tensor.

3. Spinors :

In this section, we shall define spinors and investigate their transformation laws. [1, 2].

With each point of space-time let there be associated a 2-dimensional complex vector space S, where the vectors are transformed by unimodular linear transformations:

$$T^{A} \rightarrow L^{A}{}_{B}T^{B}, \tag{1.2}$$

$$L = \det(L^{A}{}_{B}) = 1.$$

This space is called spin-space, and the vectors are called contravariant 1-spinors, or just spinors.

An element in the complex conjugate space of S is distinguished by placing a dot over the index,

$$T^{\dot{W}} \varepsilon \overline{S};$$

in particular,

$$\overline{T}^{\dot{W}} \equiv \overline{T^{W}}.$$

The transformation (1.2) induces the transformation

$$T^{\dot{W}} \rightarrow \overline{L}^{\dot{W}}{}_{\dot{X}} T^{\dot{X}}$$

in $\overline{S}$.

Let T be the dual space of S, and let P_A be any element of T, then P_A and P^A are related by the mapping

$$P_A = P^B \varepsilon_{BA}. \qquad (1.3)$$

Eq. (1.3) is equivalent to

$$P^A = \varepsilon^{AB} P_B. \qquad (1.4)$$

P_A is called a covariant 1-spinor.

If $\ell_A{}^B$ is defined by the relationship

$$\ell_A{}^B L^A{}_C = \delta^B_C,$$

which is equivalent to

$$L^C{}_B \ell_A{}^B = \delta^C_A,$$

then the transformation (1.2) induces the transformation

$$P_A \rightarrow \ell_A{}^B P_B \qquad (1.5)$$

in T.

$P_{\dot{W}}$ represents an element of $\overline{T}$. The relation (1.5) induces the transformation

$$P_{\dot{W}} \to \overline{\ell}_{\dot{W}}{}^{\dot{X}} P_{\dot{X}}$$

in $\overline{T}$.

Spinors of higher orders are obtained by considering various cartesian cross products of S, $\overline{S}$, T, and $\overline{T}$. For example, $T^{A}{}_{B\dot{X}}{}^{\dot{W}}$ denotes an element of S X T X $\overline{T}$ X $\overline{S}$. These higher order spinors transform in the same way as products of one-index spinors with the same indices;

$$T^{A\dot{W}\ldots}{}_{B\dot{X}\ldots} \to L^{A}{}_{C}\overline{L}^{\dot{W}}{}_{\dot{Y}}\ell_{B}{}^{D}\overline{\ell}_{\dot{X}}{}^{\dot{Z}}T^{C\dot{Y}\ldots}{}_{D\dot{Z}\ldots}$$

Spinors with P indices of the same kind (either dotted or undotted) are called P-spinors. A spinor with P indices of one kind and Q indices of the other is called a PQ-spinor. A PP-spinor is Hermitian if it is equal to its complex conjugate transpose:

$$T^{A_1\ldots A_p\dot{X}_1\ldots\dot{X}_p} = \overline{T^{X_1\ldots X_p\dot{A}_1\ldots\dot{A}_p}} \equiv \overline{T}^{A_1\ldots A_p\dot{X}_1\ldots\dot{X}_p}.$$

4. Spinor Algebra and Introduction of a Dyad:

We shall develop some spinor algebra, introduce a dyad into spin-space, and define dyad components. Dyad transformations will also be discussed.

From Eqs. (1.3) and (1.4), we have

$$P_A S^A = P^C \varepsilon_{CA} S^A = -P^A S_A;$$

therefore, if

$$P_A = \lambda S_A,$$

then

$$P_A S^A = 0. \tag{1.6}$$

The Levi-Civita symbol satisfies the equation

$$\varepsilon_{A[B}\varepsilon_{CD]} = 0. \tag{1.7}$$

Contracting Eq. (1.7) with an arbitrary 2-spinor T^{CD}, implies

$$T_{[AB]} = 1/2\ \varepsilon_{AB}T_C{}^C,$$

and hence

$$T_{AB} = T_{(AB)} + 1/2\ \varepsilon_{AB}T_C{}^C. \tag{1.8}$$

If

$$T_{AB} = P_A S_B,$$

then from Eq. (1.8), we obtain

$$P_A S_B - S_A P_B = \varepsilon_{AB}P_C S^C. \tag{1.9}$$

Therefore, two spinors P_A and S_A are proportional, if and only if

$$P_A S^A = 0.$$

We can take any two spinors k_A and m_A, which are not proportional, to form a basis for spin-space. If k_A and m_A are normalized so that

$$k_A m^A = 1,$$

then Eq. (1.9) gives the fundamental relationship

$$k_A m_B - m_A k_B = \varepsilon_{AB}\ , \tag{1.10}$$

which is equivalent to

$$k_A m^B - m_A k^B = \delta^B_A.$$

With this basis, we introduce the generic symbols

$$\zeta_a{}^A \equiv \{k^A,\ m^A\},$$

and

$$\bar{\zeta}_{\dot{x}}{}^{\dot{X}} \equiv \{\bar{k}^{\dot{X}}, \bar{m}^{\dot{X}}\}.$$

Eq. (1.10) implies that

$$\varepsilon^{AB} = \varepsilon^{ab}\zeta_a{}^A\zeta_b{}^B$$

if and only if

$$\varepsilon_{ab} = \zeta_a{}^A\zeta_b{}^B\varepsilon_{AB}. \tag{1.11}$$

ε_{ab} has the same numerical values as ε_{AB}.

This spinor basis, $\zeta_a{}^A$, is called a dyad. The "dyad indices" are raised and lowered with the aid of ε_{ab} as follows:

$$\zeta^{aA} = \varepsilon^{ab}\zeta_b{}^A.$$

This is equivalent to

$$\zeta_b{}^A = \zeta^{aA}\varepsilon_{ab}.$$

Explicitly, we have

$$\zeta^{1A} = \zeta_2{}^A = m^A$$

and

$$\zeta^{2A} = -\zeta_1{}^A = -k^A.$$

Any spinor, ℓ_A, can be written as

$$\ell_A = \ell^1 k_A + \ell^2 m_A = \ell^a\zeta_{aA}, \tag{1.12}$$

where ℓ^1 and ℓ^2 are defined as the dyad components of ℓ_A. Using Eq. (1.11) and Eq. (1.12), we obtain

$$\ell^a = \zeta^{aA}\ell_A.$$

The dyad components of an arbitrary spinor are defined as

$$T^{ab\dot{w}\dot{x}\ldots} = \zeta^{aA}\zeta^{bB}\bar{\zeta}^{\dot{w}\dot{W}}\bar{\zeta}^{\dot{x}\dot{X}}T_{AB\dot{W}\dot{X}\ldots} \quad .$$

These dyad indices are raised and lowered, in the obvious way, with ε_{ab}.

For a unimodular dyad transformation

$$\begin{pmatrix} k'_A \\ m'_A \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} k_A \\ m_A \end{pmatrix}, \quad ad - bc = 1, \tag{1.13}$$

the normalization of the dyad is preserved:

$$k'_A m'^A = 1.$$

The Matrix, A, in Eq. (1.13) can be decomposed into the product of three matrices as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix}.$$

Therefore, the transformation (1.13) can be generated from a transformation of the form

$$\begin{pmatrix} k'_A \\ m'_A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} k_A \\ m_A \end{pmatrix}, \tag{1.14}$$

(called a null rotation about k^A), a null rotation about m^A

$$\begin{pmatrix} k'_A \\ m'_A \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} k_A \\ m_A \end{pmatrix}, \tag{1.15}$$

and a transformation of the form

$$\begin{pmatrix} k'_A \\ m'_A \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix}\begin{pmatrix} k_A \\ m_A \end{pmatrix}. \tag{1.16}$$

These transformations [Eqs. (1.14) - (1.16)] will be interpreted, in tensor form, in the next section.

5. Correspondence Between Tensors and Spinors:

In this section, we shall give the correspondence between spinors and tensors and shall list some important spinor equivalents.

First, let us note the following conventions:

$$\overline{\delta_{X}^{W}} = \bar{\delta}_{\dot{X}}^{\dot{W}} \equiv \delta_{\dot{X}}^{\dot{W}}$$

and

$$\overline{\varepsilon^{WX}} = \bar{\varepsilon}^{\dot{W}\dot{X}} \equiv \varepsilon^{\dot{W}\dot{X}}.$$

The connecting quantities $\sigma_{\alpha}{}^{B\dot{X}}$, between spinors and tensors, behave like a space-time vector with respect to the tensor index α, and like a spinor with respect to $B\dot{X}$. These connections are Hermitian in the spinor indices

$$\sigma_{\alpha}{}^{B\dot{X}} = \bar{\sigma}_{\alpha}{}^{B\dot{X}}, \tag{1.17}$$

and satisfy the relation

$$\sigma_{\alpha}{}^{A\dot{W}}\sigma^{\alpha}{}_{B\dot{X}} = \delta^{A}_{B}\delta^{\dot{W}}_{\dot{X}}. \tag{1.18}$$

Eq. (1.18) is equivalent to

$$\sigma^{\alpha}{}_{A\dot{W}}\sigma_{\beta}{}^{A\dot{W}} = \delta^{\alpha}_{\beta}. \tag{1.19}$$

In a local coordinate frame, with

$$g_{\alpha\beta} = \eta_{\alpha\beta} = \mathrm{diag}(-1,-1,-1,+1),$$

one set of σ's which satisfy Eqs. (1.17) and (1.18) is:

$$\sigma_1{}^{B\dot{X}} = 1/\sqrt{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2{}^{B\dot{X}} = 1/\sqrt{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3{}^{B\dot{X}} = 1/\sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_4{}^{B\dot{X}} = 1/\sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \qquad (1.20)$$

We shall assume that, at any point, the σ's can be transformed into the above set.

With every tensor of rank P, we associate a PP-spinor as follows:

$$T^{A\dot{W}B\dot{X}\ldots}{}_{C\dot{Y}\ldots} = \sigma_\alpha{}^{A\dot{W}}\sigma_\beta{}^{B\dot{X}}\sigma^\gamma{}_{C\dot{Y}}T^{\alpha\beta\ldots}{}_{\gamma\ldots} \qquad (1.21)$$

Using Eqs. (1.19) and (1.21), we have

$$T^{\alpha\beta\ldots}{}_{\gamma\ldots} = \sigma^\alpha{}_{A\dot{W}}\sigma^\beta{}_{B\dot{X}}\sigma_\gamma{}^{C\dot{Y}}T^{A\dot{W}B\dot{X}\ldots}{}_{C\dot{Y}\ldots}$$

The spinor formed from $T^{\alpha\beta\ldots}{}_{\gamma\ldots}$ is called the spinor equivalent of $T^{\alpha\beta\ldots}{}_{\gamma\ldots}$. The relation between a tensor and its spinor equivalent is usually abbreviated by

$$T^{\alpha\beta\ldots}{}_{\gamma\ldots} \leftrightarrow T^{A\dot{W}B\dot{X}\ldots}{}_{C\dot{Y}\ldots} .$$

Taking the complex conjugate of Eq. (1.21), and using Eq. (1.17), we obtain

$$\overline{T}^{A\dot{W}B\dot{X}\ldots}{}_{C\dot{Y}\ldots} = \sigma_\alpha{}^{A\dot{W}}\sigma_\beta{}^{B\dot{X}}\sigma^\gamma{}_{C\dot{Y}}\overline{T}^{\alpha\beta\ldots}{}_{\gamma\ldots}$$

Therefore, the spinor equivalent of a real tensor is Hermitian and conversely.

We shall list some important spinor equivalents, and briefly indicate how they may be obtained.

(i) From Eq. (1.18), we can easily see that

$$g_{\alpha\beta} \leftrightarrow \varepsilon_{AB}\varepsilon_{\dot{W}\dot{X}}, \tag{1.22}$$

and

$$\delta_{\alpha}^{\beta} \leftrightarrow \delta_{A}^{B}\delta_{\dot{W}}^{\dot{X}} .$$

(ii) Using Eq. (1.8), it can be shown that the spinor equivalent $F_{A\dot{W}B\dot{X}}$ of a real bivector $F_{\alpha\beta}$ is given by [1, 2].

$$F_{A\dot{W}B\dot{X}} = \varepsilon_{AB}\overline{\phi}_{\dot{W}\dot{X}} + \varepsilon_{\dot{W}\dot{X}}\phi_{AB}, \tag{1.23}$$

where ϕ_{AB} is a symmetric 2-spinor.

(iii) The Riemann tensor, $R_{\alpha\beta\gamma\delta}$, satisfies the equations

$$R_{\alpha\beta\gamma\delta} = R_{[\alpha\beta][\gamma\delta]} = R_{\gamma\delta\alpha\beta} \tag{1.24}$$

and

$$R_{\alpha[\beta\gamma\delta]} = 0. \tag{1.25}$$

It can be shown that the spinor equivalent $R_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}}$ of $R_{\alpha\beta\gamma\delta}$ is given by [1, 2]

$$R_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}} = \Psi_{ABCD}\varepsilon_{\dot{W}\dot{X}}\varepsilon_{\dot{Y}\dot{Z}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{W}\dot{X}\dot{Y}\dot{Z}} + \varepsilon_{AB}\varepsilon_{\dot{Y}\dot{Z}}\Phi_{CD\dot{W}\dot{X}} + \varepsilon_{CD}\varepsilon_{\dot{W}\dot{X}}\Phi_{AB\dot{Y}\dot{Z}}$$
$$+ 2\Lambda(\varepsilon_{AC}\varepsilon_{BD}\varepsilon_{\dot{W}\dot{X}}\varepsilon_{\dot{Y}\dot{Z}} + \varepsilon_{AB}\varepsilon_{CD}\varepsilon_{\dot{W}\dot{Z}}\varepsilon_{\dot{X}\dot{Y}}). \tag{1.26}$$

As a result of Eqs. (1.24) and (1.25), the spinors in Eq. (1.26) have the following properties:

$$\Psi_{ABCD} = \Psi_{(ABCD)}$$

$$\Phi_{AB\dot{W}\dot{X}} = \Phi_{(AB)(\dot{W}\dot{X})} = \overline{\Phi}_{AB\dot{W}\dot{X}} \tag{1.27}$$

$$\Lambda = \overline{\Lambda};\ \Lambda = R/24.$$

(iv) If $R_{\alpha\beta\gamma\delta}$ also satisfies the equation

$$R_{\alpha\beta}{}^{\beta}{}_{\delta} = 0, \tag{1.28}$$

then contracting Eq. (1.26) with $\varepsilon^{BC}\varepsilon^{\dot{X}\dot{Y}}$, and symmetrizing on A and D, will imply that

$$R_{A\dot{W}B\dot{X}C\dot{Y}D\dot{Z}} = \Psi_{ABCD}\varepsilon_{\dot{W}\dot{X}}\varepsilon_{\dot{Y}\dot{Z}} + \varepsilon_{AB}\varepsilon_{CD}\overline{\Psi}_{\dot{W}\dot{X}\dot{Y}\dot{Z}}. \tag{1.29}$$

A tensor satisfying Eqs. (1.24), (1.25), and (1.28) is said to have the symmetries of the Weyl tensor. The corresponding spinor, Ψ_{ABCD}, is called a Weyl spinor.

(v) The spinor equivalent $\varepsilon_{A\dot{W}B\dot{X}}{}^{C\dot{Y}D\dot{Z}}$ of the alternating tensor $\varepsilon_{\alpha\beta\gamma\delta}$ is given by

$$\varepsilon_{A\dot{W}B\dot{X}}{}^{C\dot{Y}D\dot{Z}} = i(\delta_A^C\delta_B^D\delta_{\dot{W}}^{\dot{Z}}\delta_{\dot{X}}^{\dot{Y}} - \delta_A^D\delta_B^C\delta_{\dot{W}}^{\dot{Y}}\delta_{\dot{X}}^{\dot{Z}}). \tag{1.30}$$

Eq. (1.30) can be obtained from Eq. (1.24) and the additional requirement that

$$R_{\alpha(\beta\gamma)\delta} = 0.$$

(vi) Eqs. (1.23) and (1.30) imply that

$$F^{*}_{\alpha\beta} = 1/2\ \varepsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta} \leftrightarrow i(\varepsilon_{AB}\overline{\phi}_{\dot{W}\dot{X}} - \varepsilon_{\dot{W}\dot{X}}\phi_{AB}).$$

(vii) The tracefree Ricci tensor is defined as

$$S_{\alpha\beta} = R_{\alpha\beta} - 1/4\ Rg_{\alpha\beta}. \tag{1.31}$$

From Eqs. (1.26) and (1.22), we have

$$S_{\alpha\beta} \leftrightarrow -2\Phi_{AB\dot{W}\dot{X}}. \tag{1.32}$$

(viii) The spinor equivalent $T_{A\dot{W}}$ of a real vector T_α will be discussed in Chapter 3. In particular, if T_α is null, then

$$T_\alpha \leftrightarrow \pm T_A \overline{T}_{\dot{W}}.$$

With the dyad, $\zeta_a{}^A$, we can associate a null tetrad $\{k^\alpha, m^\alpha, t^\alpha, \overline{t}^\alpha\}$ in the space of vectors R_4, through the correspondence

$$\begin{aligned} k^\alpha &\leftrightarrow k^A \overline{k}^{\dot{W}} \\ m^\alpha &\leftrightarrow m^A \overline{m}^{\dot{W}} \\ t^\alpha &\leftrightarrow k^A \overline{m}^{\dot{W}} \\ \overline{t}^\alpha &\leftrightarrow m^A \overline{k}^{\dot{W}}. \end{aligned} \tag{1.33}$$

The only non-zero scalar products are

$$k_\alpha m^\alpha = -t_\alpha \overline{t}^\alpha = 1.$$

Using the correspondence (1.33), we see that the transformation Eq. (1.14) leaves the k^α direction invariant, and Eq. (1.15) leaves the m^α direction invariant. By letting $a = \lambda e^{i\theta/2}$ in Eq. (1.16), it is easily seen from (1.33) that this represents a Lorentz transformation in the k^α - m^α plane, and a rotation in the t^α - $\overline{t}^\alpha$ plane.

Every unimodular transformation in spin-space induces a unique Lorentz transformation in R_4. However, for each tensor index

there corresponds two spinor indices, (one dotted and one undotted), and hence the transformation $-L_A{}^B$ in S corresponds to the same transformation in R_4 as $+L_A{}^B$. Therefore, there is a 2-1 correspondence between the proper homogeneous Lorentz group in R_4, and the group of unimodular linear transformations in S.

This 2-1 relationship can easily be seen from the correspondence (1.33) by noting that the null tetrad is invariant when the matrix A in Eq. (1.13) is replaced by -A.

6. Spinor Analysis:

In this final section, we shall introduce the spinor equivalent of the covariant derivative, define spin coefficients, and translate some important tensor differential relations into spinor form.

The spinor equivalent of the covariant derivative, ∇_α, is given by the Hermitian spinor, $\nabla_{A\dot{W}}$. $\nabla_{A\dot{W}}$ is an operator which can be considered as an element of T X $\overline{T}$. The dyad, $\zeta_a{}^A$, induces the basis,

$$\{m_A\overline{m}_{\dot{W}},\ m_A\overline{k}_{\dot{W}},\ k_A\overline{m}_{\dot{W}},\ k_A\overline{k}_{\dot{W}}\}, \tag{1.34}$$

in T X $\overline{T}$. We can then write,

$$\nabla_{A\dot{W}} = m_A\overline{m}_{\dot{W}}D - m_A\overline{k}_{\dot{W}}\delta - k_A\overline{m}_{\dot{W}}\overline{\delta} + k_A\overline{k}_{\dot{W}}\Delta, \tag{1.35}$$

where D, δ, $\overline{\delta}$, and Δ are the directional derivatives, given by

$$D = k^A\overline{k}^{\dot{W}}\nabla_{A\dot{W}} = k^\alpha\nabla_\alpha$$

$$\delta = k^A\overline{m}^{\dot{W}}\nabla_{A\dot{W}} = t^\alpha\nabla_\alpha$$

$$\bar{\delta} = m^A \bar{k}^{\dot{W}} \nabla_{A\dot{W}} = \bar{t}^\alpha \nabla_\alpha$$

$$\Delta = m^A \bar{m}^{\dot{W}} \nabla_{A\dot{W}} = m^\alpha \nabla_\alpha .$$

Clearly, D and Δ are real operators.

The spin coefficients κ, σ, ρ, τ, ε, β, α, γ, π, μ, λ, υ are defined as the dyad components of the directional derivatives of $\zeta_a{}^A$:

$$Dk_A = \varepsilon k_A - \kappa m_A$$

$$\delta k_A = \beta k_A - \sigma m_A$$

$$\bar{\delta} k_A = \alpha k_A - \rho m_A$$

$$\Delta k_A = \gamma k_A - \tau m_A$$

$$Dm_A = \pi k_A - \varepsilon m_A$$

$$\delta m_A = \mu k_A - \beta m_A$$

$$\bar{\delta} m_A = \lambda k_A - \alpha m_A$$

$$\Delta m_A = \upsilon k_A - \gamma m_A \ .$$

The spin coefficients can also be defined as the dyad components of $\nabla_{C\dot{Y}} \zeta_{aB}$;

$$\Gamma_{abc\dot{y}} \equiv \zeta_b{}^B \zeta_c{}^C \bar{\zeta}_{\dot{y}}{}^{\dot{Y}} \nabla_{C\dot{Y}} \zeta_{aB} .$$

We shall list some of the important differential relationships in spinor form:

(i) The commutator relation

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)\phi = 0$$

is equivalent to

$$(\nabla_{A\dot{W}}\nabla_{B\dot{X}} - \nabla_{B\dot{X}}\nabla_{A\dot{W}})\phi = 0 \tag{1.36}$$

(ii) The Ricci identity becomes

$$\nabla_{H(\dot{W}}\nabla^{H}{}_{\dot{X})}T_D = \Phi_{DE\dot{W}\dot{X}}T^E \tag{1.37}$$

and

$$\nabla_{(A}{}^{\dot{P}}\nabla_{B)\dot{P}}T_C = -\Psi_{ABCD}T^D + 2\Lambda T_{(A}\varepsilon_{B)C} \,. \tag{1.38}$$

(iii) The Bianchi identities are

$$\nabla^{D}{}_{\dot{Y}}\Psi_{ABCD} = \nabla_{(C}{}^{\dot{Z}}\Phi_{AB)\dot{Y}\dot{Z}} \tag{1.39}$$

and

$$\nabla^{B\dot{Z}}\Phi_{AB\dot{Y}\dot{Z}} = -3\nabla_{A\dot{Y}}\Lambda. \tag{1.40}$$

(iv) Maxwell's equations, in spinor form, is the equation

$$\nabla^{B}{}_{\dot{W}}\phi_{AB} = -\ 1/2\ S_{A\dot{W}}. \tag{1.41}$$

Using Eq. (1.35), and writing Ψ_{ABCD} and $\Phi_{AB\dot{W}\dot{X}}$ in terms of their dyad components, (using the Newman-Penrose conventions), equations (1.36) to (1.41) can be written in terms of these dyad components, the directional derivatives, and the spin coefficients. These equations can be found in references [1, 3].

CHAPTER II

CLASSIFICATION OF THE WEYL TENSOR BY THE SPINOR METHOD

In this chapter, we shall classify the Weyl tensor, $C_{\alpha\beta\gamma\delta}$, by the spinor method. From Eq. (1.29), we see that a Weyl tensor determines a Weyl spinor and conversely. We shall classify the Weyl tensor, by classifying the Weyl spinor, Ψ_{ABCD}.

The Weyl spinor can be written in terms of the basis,

$$\{m_A m_B m_C m_D,\ k_{(A}m_B m_C m_{D)},\ k_{(A}k_B m_C m_{D)},\ k_{(A}k_B k_C m_{D)},\ k_A k_B k_C k_D\},$$

as

$$\Psi_{ABCD} = \psi_{abcd}\zeta^a{}_{(A}\zeta^b{}_B\zeta^c{}_C\zeta^d{}_{D)}, \tag{2.1}$$

which is equivalent to

$$\psi_{abcd} = \zeta_a{}^A\zeta_b{}^B\zeta_c{}^C\zeta_d{}^D\Psi_{ABCD}.$$

Using the notation of Newman and Penrose [1, 3], Eq. (2.1) becomes

$$\Psi_{ABCD} = \psi_0 m_A m_B m_C m_D - 4\psi_1 k_{(A}m_B m_C m_{D)} + 6\psi_2 k_{(A}k_B m_C m_{D)} - 4\psi_3 k_{(A}k_B k_C m_{D)} + \psi_4 k_A k_B k_C k_D. \tag{2.2}$$

We then have:

$$\psi_0 = \psi_{1111} = k^A k^B k^C k^D \Psi_{ABCD}$$

$$\psi_1 = \psi_{1112} = k^A k^B k^C m^D \Psi_{ABCD}$$

$$\psi_2 = \psi_{1122} = k^A k^B m^C m^D \Psi_{ABCD}$$

$$\psi_3 = \psi_{1222} = k^A m^B m^C m^D \Psi_{ABCD}$$

$$\psi_4 = \psi_{2222} = m^A m^B m^C m^D \Psi_{ABCD}.$$

We shall identify Ψ_{ABCD}, by its components;

$$(\psi_0,\ \psi_1,\ \psi_2,\ \psi_3,\ \psi_4).$$

The transformations of the dyad components for the transformation equations (1.14), (1.15), and (1.16) can be found in reference [4].

For the classification, we shall consider the expression

$$J \equiv \Psi_{ABCD}\ell^A\ell^B\ell^C\ell^D,$$

where

$$\ell^A = \ell^1 k^A + \ell^2 m^A,\ \ell^2 \neq 0,$$

is an arbitrary 1-spinor. In dyad components, we have

$$\begin{aligned} J &= \psi_{abcd}\ell^a\ell^b\ell^c\ell^d \\ &= (\psi_0 c^4 + 4\psi_1 c^3 + 6\psi_2 c^2 + 4\psi_3 c + \psi_4)(\ell^2)^4 \\ &\equiv j(c)(\ell^2)^4, \end{aligned} \tag{2.3}$$

where $c \equiv \ell^1/\ell^2$.

The quartic $j(c)$, defined in Eq. (2.3), can be factored as follows:

$$j(c) = \psi_0(c - r_1)(c - r_2)(c - r_3)(c - r_4). \tag{2.4}$$

We shall classify Ψ_{ABCD} according to the number of different factors in Eq. (2.4).

Using Eq. (2.4) in Eq. (2.3), we obtain

$$\begin{aligned} \psi_{abcd}\ell^a\ell^b\ell^c\ell^d &= [\psi_0(c - r_1)(c - r_2)(c - r_3)(c - r_4)](\ell^2)^4 \\ &= p_{(a}q_b r_c s_{d)}\ell^a\ell^b\ell^c\ell^d. \end{aligned}$$

Since ℓ^a is arbitrary, we have

$$\psi_{abcd} = p_{(a}q_b r_c s_{d)},$$

which is equivalent to

$$\Psi_{ABCD} = p_{(A}q_B r_C s_{D)}. \qquad (2.5)$$

The 1-spinors in Eq. (2.5) are determined to within constants of proportionality. If they are all distinct, then $C_{\alpha\beta\gamma\delta}$ is algebraically general. When some of the spinors are proportional, $C_{\alpha\beta\gamma\delta}$ is said to be algebraically special.

We have the following types of Weyl spinors:

Type N. $r_1 = r_2 = r_3 = r_4$, in Eq. (2.4).

From Eq. (2.5), we have

$$\Psi_{ABCD} = \pm p_A p_B p_C p_D. \qquad (2.6)$$

The plus or minus sign is determined by the sign of ψ_0. If we choose $m_A = p_A$, and k_A so that $k_A m^A = 1$, then Eq. (2.6) implies that

$$(\psi_0, 0, 0, 0, 0)$$

is the canonical form for a type N Weyl spinor.

Type III. $r_1 = r_2 = r_3 \neq r_4$.

Eq. (2.5) implies

$$\Psi_{ABCD} = p_{(A}p_B p_C q_{D)}. \qquad (2.7)$$

Choosing $m_A = p_A$ and k_A proportional to q_A, the canonical form of a type III Weyl spinor is found to be

$$(0, \psi_1, 0, 0, 0).$$

Type D. $r_1 = r_2 \neq r_3 = r_4$.

Using Eq. (2.5), we get

$$\Psi_{ABCD} = p_{(A}p_Bq_Cq_{D)}. \tag{2.8}$$

From Eq. (2.8), it is obvious that we can choose a frame in which the dyad components of a type D Weyl spinor are

$$(0,\ 0,\ \psi_2,\ 0,\ 0).$$

Type II. $r_1 = r_2 \neq r_3 \neq r_4$.

From Eq. (2.5), we obtain

$$\Psi_{ABCD} = p_{(A}p_Bq_Cr_{D)}. \tag{2.9}$$

With $m_A = p_A$, k_A proportional to q_A, and writing

$$r_A = r^1k_A + r^2m_A,$$

Eq. (2.9) gives

$$(0,\ \psi_1,\ \psi_2,\ 0,\ 0)$$

as the canonical form of Ψ_{ABCD}.

Type I. $r_1 \neq r_2 \neq r_3 \neq r_4$.

The spinors in Eq. (2.5) are all distinct. We can choose a dyad for which the Weyl spinor assumes the canonical form

$$(0,\ \psi_1,\ \psi_2,\ \psi_3,\ 0).$$

CHAPTER III

CLASSIFICATION OF A REAL VECTOR

1. Introduction:

In this chapter, we shall use the spinor method to classify a real vector. We shall first expand the spinor equivalent of this vector in terms of a basis, and then give the fundamental transformation laws for its dyad components.

2. Basis and Transformation Laws:

The spinor equivalent of a real vector, T_α, is an Hermitian spinor, $T_{A\dot{W}}$. Since $T_{A\dot{W}}$ is Hermitian, we can use the basis (1.34) to obtain

$$T_{A\dot{W}} = T_{a\dot{w}}\zeta^{a}{}_{A}\bar{\zeta}^{\dot{w}}{}_{\dot{W}}, \tag{3.1}$$

which is equivalent to

$$T_{a\dot{w}} = \zeta_{a}{}^{A}\bar{\zeta}_{\dot{w}}{}^{\dot{W}}T_{A\dot{W}}. \tag{3.2}$$

From Eq. (3.2), we see that

$$\begin{aligned} T_{1\dot{1}} &= k^{A}\bar{k}^{\dot{W}}T_{A\dot{W}} = \overline{T_{1\dot{1}}} \equiv \bar{T}_{1\dot{1}} \\ T_{1\dot{2}} &= k^{A}\bar{m}^{\dot{W}}T_{A\dot{W}} = \overline{T_{2\dot{1}}} \equiv \bar{T}_{1\dot{2}} \\ T_{2\dot{2}} &= m^{A}\bar{m}^{\dot{W}}T_{A\dot{W}} = \overline{T_{2\dot{2}}} \equiv \bar{T}_{2\dot{2}}. \end{aligned}$$

If we abbreviate $T_{1\dot{1}}$, $T_{1\dot{2}}$, and $T_{2\dot{2}}$ by T_0, T_1, and T_2, respectively, then Eq. (3.1) becomes

$$T_{A\dot{W}} = T_0 m_A\bar{m}_{\dot{W}} - T_1 m_A\bar{k}_{\dot{W}} - \bar{T}_1 k_A\bar{m}_{\dot{W}} + T_2 k_A\bar{k}_{\dot{W}}. \tag{3.3}$$

From Eq. (3.3) and the correspondence (1.33), we obtain

$$T_0 = k^A \bar{k}^{\dot{W}} T_{A\dot{W}} = k^\alpha T_\alpha$$

$$T_1 = k^A \bar{m}^{\dot{W}} T_{A\dot{W}} = t^\alpha T_\alpha$$

$$\bar{T}_1 = m^A \bar{k}^{\dot{W}} T_{A\dot{W}} = \bar{t}^\alpha T_\alpha$$

$$T_2 = m^A \bar{m}^{\dot{W}} T_{A\dot{W}} = m^\alpha T_\alpha .$$

For a different dyad, ζ'^{A}_{a},

$$T_{A\dot{W}} = T'_0 m'_A \bar{m}'_{\dot{W}} - T'_1 m'_A \bar{k}'_{\dot{W}} - \bar{T}'_1 k'_A \bar{m}'_{\dot{W}} + T'_2 k'_A \bar{k}'_{\dot{W}} . \qquad (3.4)$$

Under a null rotation about k^A, Eq. (1.14), we obtain;

$$T'_0 = T_0$$

$$T'_1 = T_0 \bar{c} + T_1$$

$$\bar{T}'_1 = T_0 c + \bar{T}_1$$

$$T'_2 = T_0 c\bar{c} + T_1 c + \bar{T}_1 \bar{c} + T_2 \qquad (3.5a)$$

For a transformation (1.15), (a null rotation about m^A), we have;

$$T'_0 = T_2 b\bar{b} + T_1 \bar{b} + \bar{T}_1 b + T_0$$

$$T'_1 = T_2 b + T_1$$

$$\bar{T}'_1 = T_2 \bar{b} + \bar{T}_1$$

$$T'_2 = T_2 . \qquad (3.5b)$$

The dyad components transform as

$$T'_0 = a\bar{a} T_0$$

$$T'_1 = a\bar{a}^{-1} T_1$$

$$\bar{T}_1' = \bar{a}a^{-1}\bar{T}_1$$

$$T_2' = a^{-1}\bar{a}^{-1}T_2,$$

for the transformation (1.16).

The invariant

$$T_\alpha T^\alpha = T_{A\dot{W}}T^{A\dot{W}} = 2(T_0T_2 - T_1\bar{T}_1), \qquad (3.6)$$

will be used to distinguish between null, space-like, and time-like vectors.

3. Classification and Canonical Forms:

$T_{A\dot{W}}$ is classified according to the sign of the invariant $T_\alpha T^\alpha$. For each class the canonical form of $T_{A\dot{W}}$ is found. We shall identify $T_{A\dot{W}}$ by its dyad components;

$$(T_0, T_1, T_2).$$

Consider the expression

$$\begin{aligned} F &\equiv \pm T_\alpha \ell^\alpha \\ &= \pm T_{A\dot{W}}\ell^A\bar{\ell}^{\dot{W}} \\ &= \pm T_{a\dot{w}}\ell^a\bar{\ell}^{\dot{w}} \\ &= \pm[T_0S\bar{S} + T_1S + \bar{T}_1\bar{S} + T_2]\ell^2\bar{\ell}^{\dot{2}} \\ &= \pm f(S)\ell^2\bar{\ell}^{\dot{2}}, \end{aligned} \qquad (3.7)$$

where $S \equiv \ell^1/\ell^2$.

We can assume that $T_0 \neq 0$. For if it is zero, a suitable null rotation about m^A will make it non-zero.

It is clear from Eq. (3.7) that f(S) vanishes, if and only if the null direction defined by ℓ^{α} is orthogonal to the vector T_{α}.

If we let $T_1 = a + ib$ and $S = x + iy$ in Eq. (3.7), we get

$$f(S) = 0$$

if and only if

$$(x + a/T_0)^2 + (y - b/T_0)^2 = -T_{A\dot{W}}T^{A\dot{W}}/2T_0. \tag{3.8}$$

Therefore, from Eq. (3.8) we see that

(i) there is one and only one null direction perpendicular to any given null vector. This direction is also parallel to the null vector.

(ii) for a space-like vector, any point on the circle (3.8) determines a null direction perpendicular to it.

(iii) no null vector is orthogonal to any time-like vector.

If $T_{A\dot{W}}$ is null, we can choose the basis spinor m_A to correspond to the null direction determined by T_{α}. From remark (i), we then obviously have

$$T_{A\dot{W}}m^{A}\bar{m}^{\dot{W}} = 0.$$

Therefore, for this dyad $T_2 = 0$ and Eq. (3.6) implies that $T_1 = 0$ also. The canonical form of the dyad components of a null vector is then

$$(T_0, 0, 0).$$

Using this dyad in spin-space, and a local cartesian coordinate system in R_4, Eq. (1.20) implies that

$$t_4 = \sigma_4{}^{B\dot{X}} T_{B\dot{X}} = 1/\sqrt{2}\ T_0.$$

Hence, $T_0 > 0$ if and only if $T_{A\dot{W}}$ corresponds to a future-pointing null vector; $T_0 < 0$ if and only if T_α is a past-pointing null vector.

If $T_{A\dot{W}}$ is space-like, then by choosing m_A and k_A to correspond to two different null directions perpendicular to $T_{A\dot{W}}$, we obtain

$$T_{A\dot{W}} m^A \bar{m}^{\dot{W}} = 0 \tag{3.9}$$

and

$$T_{A\dot{W}} k^A \bar{k}^{\dot{W}} = 0. \tag{3.10}$$

Using Eqs. (3.3), (3.9), and (3.10), we get

$$(0,\ T_1,\ 0)$$

as the canonical form of a space-like vector.

If $T_{A\dot{W}}$ is time-like, then $f(S) \neq 0$. By making a null rotation about k^A, Eq. (3.5a), with $c = -\bar{T}_1/T_0$, we have $T_0' \neq 0$, $T_1' = 0$, and $T_2' = f(c) \neq 0$. Therefore, the canonical form of a time-like vector is

$$(T_0,\ 0,\ T_2).$$

From Eq. (3.6), we must have

$$T_0 T_2 > 0.$$

In a local cartesian coordinate system, Eq. (1.20) says that

$$t_4 = \sigma_4{}^{B\dot{X}} T_{B\dot{X}} = 1/\sqrt{2}(T_0 + T_2).$$

Therefore, $T_{A\dot{W}}$ corresponds to a future-pointing time-like vector if and only if $T_0 > 0$; to a past-pointing time-like vector if and only if $T_0 < 0$.

CHAPTER IV

CLASSIFICATION OF THE TRACEFREE RICCI TENSOR

1. Introduction:

In this chapter, we shall classify the tracefree Ricci tensor. From Eq. (1.32), we see that $S_{\alpha\beta}$ determines and is determined by the tracefree Ricci spinor $\Phi_{AB\dot{W}\dot{X}}$. $\Phi_{AB\dot{W}\dot{X}}$ will be classified by the spinor method. There will be four main classes. Each will be displayed in a table at the end of Section 2. Various subclasses will be considered in Sections 3, 4, and 5. The canonical form of the tracefree Ricci spinor will be given for each subclass.

2. Classification:

In this section, the tracefree Ricci spinor will be written in terms of its dyad components; the spinor method will be used to classify $\Phi_{AB\dot{W}\dot{X}}$, and a table will be given to illustrate the various classes.

With the dyad $\zeta_a{}^A$, $\Phi_{AB\dot{W}\dot{X}}$ can be written as

$$\Phi_{AB\dot{W}\dot{X}} = \phi_{ab\dot{w}\dot{x}}\zeta^a{}_{(A}\zeta^b{}_{B)}\bar{\zeta}^{\dot{w}}{}_{(\dot{W}}\bar{\zeta}^{\dot{x}}{}_{\dot{X})}. \qquad (4.1)$$

This is equivalent to

$$\phi_{ab\dot{w}\dot{x}} = \zeta_a{}^A\zeta_b{}^B\bar{\zeta}_{\dot{w}}{}^{\dot{W}}\bar{\zeta}_{\dot{x}}{}^{\dot{X}}\Phi_{AB\dot{W}\dot{X}}.$$

Using the conventions of Newman and Penrose [1, 3], Eq. (4.1) becomes

$$\Phi_{AB\dot{W}\dot{X}} = \phi_{00}m_A m_B \overline{m}_{\dot{W}}\overline{m}_{\dot{X}} - 2\phi_{01}m_A m_B \overline{m}_{(\dot{W}}\overline{k}_{\dot{X})} + 2\phi_{02}m_A m_B \overline{k}_{\dot{W}}\overline{k}_{\dot{X}}$$

$$- 2\phi_{10}m_{(A}k_{B)}\overline{m}_{\dot{W}}\overline{m}_{\dot{X}} + 4\phi_{11}m_{(A}k_{B)}\overline{m}_{(\dot{W}}\overline{k}_{\dot{X})} - 2\phi_{12}k_{(A}m_{B)}\overline{k}_{\dot{W}}\overline{k}_{\dot{X}}$$

$$+ \phi_{20}k_A k_B \overline{m}_{\dot{W}}\overline{m}_{\dot{X}} - 2\phi_{21}k_A k_B \overline{k}_{(\dot{W}}\overline{m}_{\dot{X})} + \phi_{22}k_A k_B \overline{k}_{\dot{W}}\overline{k}_{\dot{X}}. \quad (4.2)$$

So that,

$$\phi_{00} = \phi_{11\dot{1}\dot{1}} = \overline{\phi_{00}}$$

$$\phi_{01} = \phi_{11\dot{1}\dot{2}} = \overline{\phi_{10}}$$

$$\phi_{02} = \phi_{11\dot{2}\dot{2}} = \overline{\phi_{20}}$$

$$\phi_{11} = \phi_{12\dot{1}\dot{2}} = \overline{\phi_{11}}$$

$$\phi_{12} = \phi_{12\dot{2}\dot{2}} = \overline{\phi_{21}}$$

$$\phi_{22} = \phi_{22\dot{2}\dot{2}} = \overline{\phi_{22}}.$$

Henceforth, we shall identify $\Phi_{AB\dot{W}\dot{X}}$ by its dyad components;

$$(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22}).$$

The transformations of the dyad components, for a change of dyad, can be found in Appendix I.

To classify $\Phi_{AB\dot{W}\dot{X}}$, let us consider the expression

$$G \equiv \Phi_{AB\dot{W}\dot{X}}\ell^A \ell^B \overline{\ell}^{\dot{W}} \overline{\ell}^{\dot{X}}$$

$$= - 1/2 \; S_{\alpha\beta}\ell^\alpha \ell^\beta,$$

with

$$\ell^A = \ell^1 k^A + \ell^2 m^A, \; \ell^2 \neq 0,$$

an arbitrary 1-spinor, and

$$\ell^{\alpha} \leftrightarrow \pm \ell^{A}\bar{\ell}^{\dot{W}}$$

the corresponding null vector. In terms of the dyad components

$$G = \phi_{ab\dot{w}\dot{x}}\ell^{a}\ell^{b}\bar{\ell}^{\dot{w}}\bar{\ell}^{\dot{x}}$$

$$= [\phi_{00}S^{2}\bar{S}^{2} + 2\phi_{01}S^{2}\bar{S} + 2\phi_{10}S\bar{S}^{2} + \phi_{02}S^{2} + \phi_{20}\bar{S}^{2} + 4\phi_{11}S\bar{S}$$

$$+ 2\phi_{12}S + 2\phi_{21}\bar{S} + \phi_{22}]\ \ell^{2}\ell^{2}\bar{\ell}^{\dot{2}}\bar{\ell}^{\dot{2}}$$

$$\equiv g(S)\ell^{2}\ell^{2}\bar{\ell}^{\dot{2}}\bar{\ell}^{\dot{2}}, \qquad (4.3)$$

where $S \equiv \ell^{1}/\ell^{2}$.

It is clear that

$$g(S) = \overline{g(S)}.$$

The following theorem will be proven in Appendix I:

Theorem:

The function g(S), in Eq. (4.3), can be factored:

$$g(S) = 1/2\ (T_{1\dot{1}}S\bar{S} + T_{1\dot{2}}S + T_{2\dot{1}}\bar{S} + T_{2\dot{2}})(P_{1\dot{1}}S\bar{S} + P_{1\dot{2}}S + P_{2\dot{1}}\bar{S} + P_{2\dot{2}})$$

$$+ 1/2\ (\bar{T}_{1\dot{1}}S\bar{S} + \bar{T}_{1\dot{2}}S + \bar{T}_{2\dot{1}}\bar{S} + \bar{T}_{2\dot{2}})(\bar{P}_{1\dot{1}}S\bar{S} + \bar{P}_{1\dot{2}}S + \bar{P}_{2\dot{1}}\bar{S} + P_{2\dot{2}}) \qquad (4.4)$$

$T_{a\dot{w}}$ and $P_{a\dot{w}}$ are, in general, complex numbers. The factorization is not unique.

By defining

$$T(S) = T_{1\dot{1}}S\bar{S} + T_{1\dot{2}}S + T_{2\dot{1}}\bar{S} + T_{2\dot{2}}$$

and

$$P(S) = P_{1\dot{1}}S\bar{S} + P_{1\dot{2}}S + P_{2\dot{1}}\bar{S} + P_{2\dot{2}},$$

Eq. (4.4) becomes

$$g(S) = 1/2\ [T(S)P(S) + \overline{T(S)}\,\overline{P(S)}]. \qquad (4.5)$$

Putting Eq. (4.5) in Eq. (4.3), we get

$$\phi_{ab\dot{w}\dot{x}}\ell^a\ell^b\bar{\ell}^{\dot{w}}\bar{\ell}^{\dot{x}} = 1/2\ [T(S)P(S) + \overline{T(S)}\,\overline{P(S)}]\ \ell^2\ell^2\bar{\ell}^{\dot{2}}\bar{\ell}^{\dot{2}}$$

$$= 1/2\ [T_{a\dot{w}}P_{b\dot{x}} + \bar{T}_{a\dot{w}}\bar{P}_{b\dot{x}}]\ \ell^a\ell^b\bar{\ell}^{\dot{w}}\bar{\ell}^{\dot{x}}.$$

Since ℓ^a is arbitrary, we have

$$\phi_{ab\dot{w}\dot{x}} = 1/2\ \mathcal{S}[T_{a\dot{w}}P_{b\dot{x}} + \bar{T}_{a\dot{w}}\bar{P}_{b\dot{x}}], \qquad (4.6)$$

where

$$\mathcal{S}[T_{a\dot{w}}P_{b\dot{x}}] \equiv 1/4\ (T_{a\dot{w}}P_{b\dot{x}} + T_{b\dot{w}}P_{a\dot{x}} + T_{a\dot{x}}P_{b\dot{w}} + T_{b\dot{x}}P_{a\dot{w}}).$$

(The operator $\mathcal{S}$ symmetrizes with respect to the pair of undotted indices and the pair of dotted indices).

Eq. (4.6) is equivalent to

$$\Phi_{AB\dot{W}\dot{X}} = 1/2\ \mathcal{S}[T_{A\dot{W}}P_{B\dot{X}} + \bar{T}_{A\dot{W}}\bar{P}_{B\dot{X}}], \qquad (4.7)$$

where

$$T_{A\dot{W}} \equiv T_{1\dot{1}}m_A\bar{m}_{\dot{W}} - T_{1\dot{2}}m_A\bar{k}_{\dot{W}} - T_{2\dot{1}}k_A\bar{m}_{\dot{W}} + T_{2\dot{2}}k_A\bar{k}_{\dot{W}}$$

and

$$P_{A\dot{W}} \equiv P_{1\dot{1}}m_A\bar{m}_{\dot{W}} - P_{1\dot{2}}m_A\bar{k}_{\dot{W}} - P_{2\dot{1}}k_A\bar{m}_{\dot{W}} + P_{2\dot{2}}k_A\bar{k}_{\dot{W}}.$$

Conversely, if Eq. (4.7) holds, then from Eq. (4.3)

$$\phi_{ab\dot{w}\dot{x}}\ell^a\ell^b\bar{\ell}^{\dot{w}}\bar{\ell}^{\dot{x}} = g(S)\ell^2\ell^2\bar{\ell}^{\dot{2}}\bar{\ell}^{\dot{2}}$$

$$= 1/2\ (T_{a\dot{w}}P_{b\dot{x}} + \bar{T}_{a\dot{w}}\bar{P}_{b\dot{x}})\ \ell^a\ell^b\bar{\ell}^{\dot{w}}\bar{\ell}^{\dot{x}}$$

$$= 1/2\ (T(S)P(S) + \overline{T(S)}\,\overline{P(S)})\ \ell^2\ell^2\bar{\ell}^{\dot{2}}\bar{\ell}^{\dot{2}}$$

and hence,

$$g(S) = 1/2\ (T(S)P(S) + \overline{T(S)}\,\overline{P(S)}).$$

With the aid of Eq. (1.8), Eq. (4.7) can be written as

$$\Phi_{AB\dot{W}\dot{X}} = 1/4\ (T_{A\dot{W}}P_{B\dot{X}} + \overline{T}_{A\dot{W}}\overline{P}_{B\dot{X}} + T_{B\dot{X}}P_{A\dot{W}} + \overline{T}_{B\dot{X}}\overline{P}_{A\dot{W}})$$

$$- 1/8\ \varepsilon_{AB}\varepsilon_{\dot{W}\dot{X}}(T_{C\dot{Y}}P^{C\dot{Y}} + \overline{T}_{C\dot{Y}}\overline{P}^{C\dot{Y}}). \qquad (4.8)$$

In tensor form, Eq. (4.8) becomes

$$S_{\alpha\beta} = 1/4\ g_{\alpha\beta}(T_{\gamma}P^{\gamma} + \overline{T}_{\gamma}\overline{P}^{\gamma}) - [T_{(\alpha}P_{\beta)} + \overline{T}_{(\alpha}\overline{P}_{\beta)}]. \qquad (4.9)$$

By comparing the coefficients in either Eq. (4.5) or Eq. (4.7), we get the following set of equations relating the dyad components of $\Phi_{AB\dot{W}\dot{X}}$ to those of $T_{A\dot{W}}$ and $P_{A\dot{W}}$:

$$\phi_{00} = 1/2\ (T_{1\dot{1}}P_{1\dot{1}} + \overline{T}_{1\dot{1}}\overline{P}_{1\dot{1}})$$

$$\phi_{01} = 1/4\ (T_{1\dot{1}}P_{1\dot{2}} + P_{1\dot{1}}T_{1\dot{2}} + \overline{T}_{1\dot{1}}\overline{P}_{1\dot{2}} + \overline{P}_{1\dot{1}}\overline{T}_{1\dot{2}})$$

$$\phi_{02} = 1/2\ (T_{1\dot{2}}\ P_{1\dot{2}} + \overline{T}_{1\dot{2}}\overline{P}_{1\dot{2}})$$

$$\phi_{11} = 1/8\ (T_{1\dot{1}}P_{2\dot{2}} + P_{1\dot{1}}T_{2\dot{2}} + \overline{T}_{1\dot{1}}\overline{P}_{2\dot{2}} + \overline{P}_{1\dot{1}}\overline{T}_{2\dot{2}} + T_{1\dot{2}}P_{2\dot{1}} + T_{2\dot{1}}P_{1\dot{2}}$$
$$+ \overline{T}_{1\dot{2}}\overline{P}_{2\dot{1}} + \overline{T}_{2\dot{1}}\overline{P}_{1\dot{2}})$$

$$\phi_{12} = 1/4\ (T_{1\dot{2}}P_{2\dot{2}} + P_{1\dot{2}}T_{2\dot{2}} + \overline{T}_{1\dot{2}}\overline{P}_{2\dot{2}} + \overline{P}_{1\dot{2}}\overline{T}_{2\dot{2}})$$

$$\phi_{22} = 1/2\ (T_{2\dot{2}}P_{2\dot{2}} + \overline{T}_{2\dot{2}}\overline{P}_{2\dot{2}}). \qquad (4.10)$$

We shall consider the following classes of $\Phi_{AB\dot{W}\dot{X}}$:

Class I: In Eq. (4.7), $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are Hermitian and equal.

Class II: $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are Hermitian and not proportional.

Class III: $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are non-Hermitian and are complex conjugates of each other. We shall also require that $T_{A\dot{W}}$ is not simply a product of an Hermitian spinor and a complex number.

Class IV: $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are non-Hermitian and not conjugates of each other.

The following table will be used to illustrate the equations satisfied by $\Phi_{AB\dot{W}\dot{X}}$ and g(S) in each of the above classes. Classes I, II, and III will be discussed in some detail in the following sections.

Class	Properties of $T_{A\dot{W}}$, $P_{A\dot{W}}$	Eq. Satisfied by $\Phi_{AB\dot{W}\dot{X}}$ (Eq. (4.7))	Properties of $T(S)$, $P(S)$	Eq. Satisfied by $g(S)$ (Eq. (4.5))
I	$T_{A\dot{W}} = \overline{T}_{A\dot{W}}$	$\Phi_{AB\dot{W}\dot{X}} = \pm\int(T_{A\dot{W}}T_{B\dot{X}})$	$T(S) = \overline{T(S)}$	$g(S) = \pm T(S)T(S)$
II	$T_{A\dot{W}} = \overline{T}_{A\dot{W}}$	$\Phi_{AB\dot{W}\dot{X}} = \int(T_{A\dot{W}}P_{B\dot{X}})$	$T(S) = \overline{T(S)}$	$g(S) = T(S)P(S)$
	$P_{A\dot{W}} = \overline{P}_{A\dot{W}}$		$P(S) = \overline{P(S)}$	
III	$T_{A\dot{W}} \neq \overline{T}_{A\dot{W}}$	$\Phi_{AB\dot{W}\dot{X}} = \pm\int(T_{A\dot{W}}\overline{T}_{B\dot{X}})$	$T(S) \neq \overline{T(S)}$	$g(S) = \pm T(S)\overline{T(S)}$
	$T_{A\dot{W}} = \overline{P}_{A\dot{W}}$		$T(S) = \overline{P(S)}$	
IV	$T_{A\dot{W}} \neq \overline{T}_{A\dot{W}}$	$\Phi_{AB\dot{W}\dot{X}} = 1/2 \int(T_{A\dot{W}}P_{B\dot{X}} + \overline{T}_{A\dot{W}}\overline{P}_{B\dot{X}})$	$T(S) \neq \overline{T(S)}$	$g(S) = 1/2\ (T(S)P(S) + \overline{T(S)}\,\overline{P(S)})$
	$P_{A\dot{W}} \neq \overline{P}_{A\dot{W}}$		$P(S) \neq \overline{P(S)}$	
	$T_{A\dot{W}} \neq \overline{P}_{A\dot{W}}$		$T(S) \neq \overline{P(S)}$	

3. Class I:

In this section, a class I tracefree Ricci spinor will be analysed in some detail. Six subclasses will be considered, and for each subclass the canonical form of $\Phi_{AB\dot{W}\dot{X}}$ will be given. A table will be used to illustrate these subclasses.

For a class I tracefree Ricci spinor, Eq. (4.7) reduces to

$$\Phi_{AB\dot{W}\dot{X}} = \pm\mathcal{S}(T_{A\dot{W}}T_{B\dot{X}}). \tag{4.11}$$

This is equivalent to, (see Eq. (4.5)),

$$g(S) = \pm T(S)T(S), \tag{4.12}$$

where $T(S) = \overline{T(S)}$.

The factorization Eq. (4.12) is unique up to a transformation

$$T(S) \to -T(S). \tag{4.13}$$

With $T(S) = P(S) = \overline{T(S)}$, the set of equations (4.10) reduce to:

$$\begin{aligned}
\phi_{00} &= \pm T_0^2 \\
\phi_{01} &= \pm T_0 T_1 \\
\phi_{02} &= \pm T_1^2 \\
\phi_{11} &= \pm\ 1/2\ (T_0 T_2 + T_1 \overline{T}_1) \\
\phi_{12} &= \pm T_1 T_2 \\
\phi_{22} &= \pm T_2^2 .
\end{aligned} \tag{4.14}$$

Since the factorization, Eq. (4.12), is unique up to the transformation (4.13), we see that $T_\alpha T^\alpha$ given by

$$T_\alpha T^\alpha = 2(T_0 T_2 - T_1 \overline{T}_1),$$

is determined uniquely from the set of equations (4.14).

From Eq. (4.9), we get the equation satisfied by a class I tracefree Ricci tensor;

$$S_{\alpha\beta} = \pm[1/2\ g_{\alpha\beta}T_{\gamma}T^{\gamma} - 2T_{\alpha}T_{\beta}].$$

Using the canonical forms of $T_{A\dot{W}}$, which were found in Chapter III, the following table lists the canonical forms of $\Phi_{AB\dot{W}\dot{X}}$ for the various subclasses of class I. The subclasses are determined by the plus or minus sign in the factorization Eq. (4.12), and the sign of the invariant $T_{\alpha}T^{\alpha}$. The canonical forms of $\Phi_{AB\dot{W}\dot{X}}$ are obtained immediately from the set of equations (4.14) and the canonical forms for $T_{A\dot{W}}$.

Name of Subclass	Factorization of $g(S)$	Sign of $T^2 \equiv T_\alpha T^\alpha$	Canonical Form of $T_{A\dot{W}}$ (T_0, T_1, T_2)	Canonical Form of $\Phi_{AB\dot{W}\dot{X}}$ $(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22})$
I-A.1	$g(S) = T(S)T(S)$	$T^2 = 0$	$(T_0, 0, 0)$	$(\phi_{00}, 0, 0, 0, 0, 0)$ $\phi_{00} > 0$
I-A.2	"	$T^2 < 0$	$(0, T_1, 0)$	$(0, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{11} > 0$
I-A.3	"	$T^2 > 0$	$(T_0, 0, T_2)$ $T_0T_2 > 0$	$(\phi_{00}, 0, 0, \phi_{11}, 0, \phi_{22})$ $\phi_{00}, \phi_{11}, \phi_{22} > 0$
I-B.1	$g(S) = -T(S)T(S)$	$T^2 = 0$	$(T_0, 0, 0)$	$(\phi_{00}, 0, 0, 0, 0, 0)$ $\phi_{00} < 0$
I-B.2	"	$T^2 < 0$	$(0, T_1, 0)$	$(0, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{11} < 0$
I-B.3	"	$T^2 > 0$	$(T_0, 0, T_2)$ $T_0T_2 > 0$	$(\phi_{00}, 0, 0, \phi_{11}, 0, \phi_{22})$ $\phi_{00}, \phi_{11}, \phi_{22} < 0$

4. Class II:

In this section, a class II tracefree Ricci spinor will be analysed. Various subclasses will be listed in a table at the end of the section. For each subclass, the canonical form of $\Phi_{AB\dot{W}\dot{X}}$ will be found.

With $\Phi_{AB\dot{W}\dot{X}}$ in class II, Eq. (4.7) reduces to

$$\Phi_{AB\dot{W}\dot{X}} = \mathcal{S}(T_{A\dot{W}}P_{B\dot{X}}), \qquad (4.15)$$

which is equivalent to

$$g(S) = T(S)P(S).$$

(Obviously T(S) and P(S) are real, since $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are Hermitian). This factorization is unique up to an interchange of T(S) and P(S), and a transformation

$$\begin{aligned} &T(S) \to \kappa T(S) \\ &P(S) \to 1/\kappa\ P(S), \text{ (for } \kappa \text{ real).} \end{aligned} \qquad (4.16)$$

For a class II tracefree Ricci spinor, Eq. (4.10) reduces to:

$$\begin{aligned}
\phi_{00} &= T_0P_0 \\
\phi_{01} &= 1/2\ (T_0P_1 + T_1P_0) \\
\phi_{02} &= T_1P_1 \\
\phi_{11} &= 1/4\ (T_0P_2 + P_0T_2 + \bar{T}_1P_1 + \bar{P}_1T_1) \\
\phi_{12} &= 1/2\ (T_1P_2 + P_1T_2) \\
\phi_{22} &= T_2P_2
\end{aligned} \qquad (4.17)$$

($P_{1\dot{1}}$, $P_{1\dot{2}}$ and $P_{2\dot{2}}$ are defined as P_0, P_1, and P_2, respectively).

This set of equations is invariant under an interchange of T(S) and P(S), and under a transformation (4.16). For the degrees of freedom in the factorization of g(S), the sign of $T_\alpha T^\alpha$, $P_\alpha P^\alpha$, and $T_\alpha P^\alpha$, given by

$$T_\alpha P^\alpha = T_{A\dot{W}} P^{A\dot{W}} = T_0 P_2 + P_0 T_2 - T_1 \bar{P}_1 - P_1 \bar{T}_1, \qquad (4.18)$$

is invariant.

From Eq. (4.9), we see that

$$S_{\alpha\beta} = 1/2\, g_{\alpha\beta} T_\gamma P^\gamma - 2T_{(\alpha} P_{\beta)},$$

if and only if $S_{\alpha\beta}$ is in class II.

The following table lists the various subclasses of a class II tracefree Ricci spinor. These subclasses are distinguished by invariants. The simplest forms that <u>both</u> $T_{A\dot{W}}$ and $P_{A\dot{W}}$ can assume are listed and the proofs are given in Appendix II. The canonical forms of $\Phi_{AB\dot{W}\dot{X}}$ can be found from Eq. (4.17).

The following definitions are used in the table:

$$T^2 \equiv T_\alpha T^\alpha$$

$$P^2 \equiv P_\alpha P^\alpha$$

$$TP \equiv T_\alpha P^\alpha \text{ (see Eq. (4.18))}$$

$$R \equiv (P_\alpha P^\alpha)(T_\beta T^\beta) - (P_\alpha T^\alpha)^2.$$

Name of Subclass	Invariants	Simplest Form of both $T_{A\dot{W}}$ and $P_{A\dot{W}}$	Canonical Form of $\Phi_{AB\dot{W}\dot{X}}$ $(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22})$	Comment
II-A	$T^2 = 0$ $P^2 = 0$	$(0, 0, T_2)$ $(P_0, 0, 0)$	$(0, 0, 0, \phi_{11}, 0, 0)$ $TP = 4\phi_{11} \neq 0$	$TP \neq 0$ since T_α and P_α are not proportional. $TP > 0$ iff both P_α and T_α are future-pointing or both past-pointing vectors. $TP < 0$ iff P_α is past-pointing and T_α future-pointing, or vice-versa.
II-B.1	$T^2 < 0$ $P^2 = 0$ $TP = 0$	$(0, T_1, 0)$ $(P_0, 0, 0)$	$(0, \phi_{01}, 0, 0, 0, 0)$	
II-B.2	$T^2 < 0$ $P^2 = 0$ $TP \neq 0$	$(0, T_1, T_2)$ $(P_0, 0, 0)$	$(0, \phi_{01}, 0, \phi_{11}, 0, 0)$ $TP = 4\phi_{11}$	$TP \gtrless 0$.
II-C	$T^2 > 0$ $P^2 = 0$	$(T_0, 0, T_2)$ $(P_0, 0, 0)$	$(\phi_{00}, 0, 0, \phi_{11}, 0, 0)$ $TP = 4\phi_{11} \neq 0$	$TP \gtrless 0$. $TP \neq 0$, since no time-like vector is orthogonal to a null vector.

Name of Subclass	Invariants	Simplest Form of both $T_{A\dot{W}}$ and $P_{A\dot{W}}$	Canonical Form of $\Phi_{AB\dot{W}\dot{X}}$ $(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22})$	Comment
II-D.1	$T^2 < 0$ $P^2 > 0$ $TP = 0$	$(0, T_1, 0)$ $(P_0, 0, P_2)$	$(0, \phi_{01}, 0, 0, \phi_{12}, 0)$	
II-D.2	$T^2 < 0$ $P^2 > 0$ $TP \neq 0$	$(0, T_1, 0)$ (P_0, P_1, P_2)	$(0, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, 0)$ $TP = -4\phi_{11}$	
II-E.1	$T^2 < 0$ $P^2 < 0$ $R > 0$	$(0, T_1, 0)$ $(0, P_1, 0)$	$(0, 0, \phi_{02}, \phi_{11}, 0, 0)$ $TP = -4\phi_{11}$	$TP \lesseqgtr 0$.
II-E.2	$T^2 < 0$ $P^2 < 0$ $R = 0$	$(0, T_1, 0)$ $(P_0, P_1, 0)$	$(0, \phi_{01}, \phi_{02}, \phi_{11}, 0, 0)$ $TP = -4\phi_{11} \neq 0$	For $R = 0$, we must have $TP \neq 0$.
II- E.3	$T^2 < 0$ $P^2 < 0$ $TP < 0$	$(0, T_1, 0)$ (P_0, P_1, P_2)	$(0, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, 0)$ $TP = -4\phi_{11} \neq 0$	For $R < 0$, then TP cannot vanish.

Name of Subclass	Invariants	Simplest Form of both $T_{A\dot{W}}$ and $P_{A\dot{X}}$	Canonical Form of $\Phi_{AB\dot{W}\dot{X}}$ $(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22})$	Comment
II-F	$T^2 > 0$ $P^2 > 0$	$(T_0, 0, T_2)$ $(P_0, 0, P_2)$	$(\phi_{00}, 0, 0, \phi_{11}, 0, \phi_{22})$ $TP = 4\phi_{11} \neq 0$	$TP \neq 0$ since two time-like vectors cannot be perpendicular. $TP > 0$ iff T_α and P_α are both past-pointing, or both future-pointing vectors. $TP < 0$ iff T_α is past-pointing and P_α future-pointing, or vice versa.

5. Class III:

A class III tracefree Ricci spinor satisfies the equation

$$\Phi_{AB\dot{W}\dot{X}} = \pm\int(T_{A\dot{W}}\overline{T}_{B\dot{X}}). \tag{4.19}$$

We require that $T_{A\dot{W}}$ is not simply a product of a complex number times an Hermitian spinor, for if this were the case, Eq. (4.19) would reduce to Eq. (4.11) and then $\Phi_{AB\dot{W}\dot{X}}$ would also be in class I.

Since $T_{A\dot{W}} = \overline{P}_{A\dot{W}}$, Eq. (4.5) becomes

$$g(S) = \pm T(S)\overline{T(S)}. \tag{4.20}$$

This factorization is unique up to an interchange of T(S) and $\overline{T(S)}$, and a transformation

$$T(S) \to e^{i\theta}T(S), \tag{4.21}$$

for any real θ. This transformation is called a duality rotation.

T(S) can be written as

$$T(S) = R(S) + iQ(S), \tag{4.22}$$

where R(S) and Q(S) are real.

Using Eq. (4.22) in Eq. (4.20), we have

$$g(S) = \pm[R(S)R(S) + Q(S)Q(S)], \tag{4.23}$$

and therefore

$$\Phi_{AB\dot{W}\dot{X}} = \pm\int(R_{A\dot{W}}R_{B\dot{X}} + Q_{A\dot{W}}Q_{B\dot{X}}). \tag{4.24}$$

Both $R_{A\dot{W}}$ and $Q_{A\dot{W}}$ in Eq. (4.24) are obviously Hermitian.

Using Eq. (4.4), or by comparing coefficients in Eq. (4.23), we obtain:

$$\phi_{00} = \pm[R_0^2 + Q_0^2]$$

$$\begin{aligned}
\phi_{01} &= \pm[R_0R_1 + Q_0Q_1] \\
\phi_{02} &= \pm[R_1^2 + Q_1^2] \\
\phi_{11} &= \pm\ 1/2\ [R_0R_2 + Q_0Q_2 + R_1\bar{R}_1 + Q_1\bar{Q}_1] \\
\phi_{12} &= \pm[R_1R_2 + Q_1Q_2] \\
\phi_{22} &= \pm[R_2^2 + Q_2^2].
\end{aligned} \tag{4.25}$$

(R_0, R_1, and R_2 are defined as $R_{1\dot{1}}$, $R_{1\dot{2}}$, and $R_{2\dot{2}}$ respectively; and similarly for Q_0, Q_1, and Q_2).

From Eqs. (4.22) and (4.9), we get

$$S_{\alpha\beta} = \pm[1/2\ g_{\alpha\beta}(R_\gamma R^\gamma)(Q_\delta Q^\delta) - 2(R_\alpha R_\beta + Q_\alpha Q_\beta)].$$

Because of the freedom of a duality rotation in the factorization of g(S), we can assume, without loss of generality, that $R_{A\dot{W}}$ and $Q_{A\dot{W}}$ are orthogonal. For if $R_\alpha Q^\alpha \neq 0$, we can make a duality rotation (4.21) and obtain

$$R'_\alpha Q'^\alpha = 1/2\ (R_\alpha R^\alpha - Q_\alpha Q^\alpha)\ \sin 2\theta + R_\alpha Q^\alpha \cos 2\theta.$$

By choosing

$$\theta = 1/2\ \operatorname{arccot}\{(Q_\alpha Q^\alpha - R_\alpha R^\alpha)/R_\alpha Q^\alpha\},$$

we get

$$R'_\alpha Q'^\alpha = 0.$$

In view of what was said in Chapter III, with R_α and Q_α orthogonal, we need to consider only the following cases:

(i) R_α space-like, Q_α null.

(ii) R_α space-like, Q_α space-like.

(iii) R_α space-like, Q_α time-like.

These cases can be distinguished by an invariant. Let us define

$$I \equiv (T_\alpha \bar{T}^\alpha)^2 - (T_\alpha T^\alpha)(\bar{T}_\beta \bar{T}^\beta), \tag{4.26}$$

where $T_\alpha = R_\alpha + iQ_\alpha$. (See Eq. (4.22)). Then, since R_α and Q_α are perpendicular,

$$I = 4(R_\alpha R^\alpha)(Q_\beta Q^\beta).$$

Therefore, for the above three cases, we have

$$\text{(i) } I = 0 \text{ iff } T^2 < 0,\ P^2 = 0$$

$$\text{(ii) } I > 0 \text{ iff } T^2 < 0,\ P^2 < 0$$

$$\text{(iii) } I < 0 \text{ iff } T^2 < 0,\ P^2 > 0.$$

($T^2 \equiv T_\alpha T^\alpha$ and $P^2 \equiv P_\alpha P^\alpha$).

From the definition (4.26), we can see that "I" is invariant under a duality rotation and an interchange of T(S) and $\overline{T(S)}$.

The following table lists the various subclasses. The reduction of $R_{A\dot{W}}$ and $Q_{A\dot{W}}$ to their simplest forms can be obtained by considering the corresponding subclasses of class II. The canonical forms of $\Phi_{AB\dot{W}\dot{X}}$ can be read off from the set of equations (4.25).

Name of Subclass	Factorization of g(S)	Invariant	Simplest Form of both $R_{A\dot{W}}$ and $Q_{A\dot{W}}$	Canonical Form of $\Phi_{AB\dot{W}\dot{X}}$ $(\phi_{00}, \phi_{01}, \phi_{02}, \phi_{11}, \phi_{12}, \phi_{22})$
III-A.1	$g(S) = T(S)\overline{T(S)}$	$I = 0$	$(0, R_1, 0)$ $(Q_0, 0, 0)$	$(\phi_{00}, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{00}, \phi_{11} > 0$
III-A.2	"	$I > 0$	$(0, R_1, 0)$ $(0, Q_1, 0)$	$(0, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{11} > 0$
III-A.3	"	$I < 0$	$(0, R_1, 0)$ $(Q_0, 0, Q_2)$	$(\phi_{00}, 0, \phi_{02}, \phi_{11}, 0, \phi_{22})$ $\phi_{00}, \phi_{11}, \phi_{22} > 0$
III-B.1	$g(S) = -T(S)\overline{T(S)}$	$I = 0$	$(0, R_1, 0)$ $(Q_0, 0, 0)$	$(\phi_{00}, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{00}, \phi_{11} < 0$
III-B.2	"	$I > 0$	$(0, R_1, 0)$ $(0, Q_1, 0)$	$(0, 0, \phi_{02}, \phi_{11}, 0, 0)$ $\phi_{11} < 0$
III-B.3	"	$I < 0$	$(0, R_1, 0)$ $(Q_0, 0, Q_2)$	$(\phi_{00}, 0, \phi_{02}, \phi_{11}, 0, \phi_{22})$ $\phi_{00}, \phi_{11}, \phi_{22} < 0$

CHAPTER V

AN APPLICATION OF THE CLASSIFICATION OF THE TRACEFREE RICCI TENSOR

1. Introduction:

This chapter deals with an application of the classification given in Chapter IV. In Section 2, we find necessary and sufficient conditions for the geometry of space-time to have as its source a real scalar field. In Section 3, these conditions are related to those obtained by D. R. Brill [5] and R. Penney [6].

2. Real Scalar Field:

The purpose of Geometrodynamics is to describe non-gravitational physical fields, as, for example, the electromagnetic field, in geometric terms [7, 8]. The three basic approaches towards this goal are: the non-Riemannian theories, the five-dimensional theories, and the "already unified" field theory.

We shall now consider an example of the "already unified" field theory by finding necessary and sufficient conditions on the geometry of space-time to have as its source a real scalar field [5, 6].

If $\phi(x)$ denotes the scalar field, then the Lagrangian is given by

$$L = 1/2\ g^{\alpha\beta}\nabla_{\alpha}\phi\nabla_{\beta}\phi.$$

From this we may easily deduce the equation of motion

$$g^{\alpha\beta}\nabla_\alpha(\nabla_\beta\phi) = 0, \tag{5.1}$$

and the energy-momentum tensor

$$T_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - g_{\alpha\beta}L. \tag{5.2}$$

Using Eq. (5.2) and Eq. (1.1), we obtain

$$R_{\alpha\beta} = -\nabla_\alpha\phi\nabla_\beta\phi. \tag{5.3}$$

By defining

$$t_\alpha = \nabla_\alpha\phi, \tag{5.4}$$

Eq. (5.3) becomes

$$R_{\alpha\beta} = -t_\alpha t_\beta, \tag{5.5}$$

and hence

$$S_{\alpha\beta} = -t_\alpha t_\beta + 1/4\ g_{\alpha\beta}t_\gamma t^\gamma. \tag{5.6a}$$

If we define $p_\alpha = 1/\sqrt{2}\ t_\alpha$, then Eq. (5.6a) becomes

$$S_{\alpha\beta} = -2p_\alpha p_\beta + 1/2\ g_{\alpha\beta}p_\gamma p^\gamma \tag{5.6b}$$

and therefore, $S_{\alpha\beta}$ is in class I-A.

Conversely, if $S_{\alpha\beta}$ is in class I-A, then we can find a vector p_α which satisfies Eq. (5.6b). If we also require that

$$R = -2p_\alpha p^\alpha \equiv -t_\alpha t^\alpha, \tag{5.7}$$

then obviously Eq. (5.5) is satisfied.

A necessary and sufficient condition that there exist a function $\phi(x)$ satisfying Eq. (5.4) is

$$\nabla_{[\alpha}t_{\beta]} = 0. \tag{5.8}$$

This is clearly a geometrical condition, since t_α is uniquely

determined (up to sign) by $S_{\alpha\beta}$. t_α must also satisfy the differential equation (5.1). However, from the Bianchi identity (Eq. (1.40)), Eqs. (5.7) and (5.8) we can easily see that Eq. (5.1) is a consequence of the geometry produced by the scalar field.

In conclusion, we then have that $R_{\alpha\beta}$ satisfies Eq. (5.3) if and only if $S_{\alpha\beta}$ is in class I-A, and the vector t_α determined by $S_{\alpha\beta}$ satisfies Eq. (5.7) and the compatibility conditions (5.8).

3. Comparison of Tensor Conditions With Those Found in Section 2:

The following necessary and sufficient conditions for $R_{\alpha\beta}$ to satisfy Eq. (5.3) were found by R. Penney [6]:

$$R_\alpha{}^\gamma R_{\gamma\beta} = RR_{\alpha\beta} \tag{5.9}$$

$$G_{\alpha\beta}V^\alpha V^\beta \leq 0, \tag{5.10}$$

for arbitrary time-like vectors V^α, ($G_{\alpha\beta}$ is the Einstein tensor),

$$R>0 \tag{5.11}$$

and

$$R^{\delta\gamma}\nabla_{[\beta}\nabla_{\alpha]}R_\gamma - 1/2\ R^\delta{}_{[\alpha}\nabla_{\beta]}R = 0. \tag{5.12}$$

We shall see that Eqs. (5.9) and (5.10) imply that $S_{\alpha\beta}$ is in class I-A, and that R is determined by $S_{\alpha\beta}$ as in Eq. (5.7).

The condition (5.11) is not necessary and is not required for the physical situation as suggested by Penney. In Section 2, we did not restrict the sign of R. Also, Eq. (5.12) is identically zero when R = 0, i.e. when t_α in Eq. (5.7) is null. D. R. Brill [5]

also obtained Eqs. (5.9) and (5.12) for the geometrization, but neglected to mention Eq. (5.10). In his proof, he implicitly assumed that R was positive.

We will now translate Eq. (5.9) into spinor form. Using the spinor equivalents of $S_{\alpha\beta}$, $R_{\alpha\beta}$, and $g_{\alpha\beta}$, we get

$$4\phi_{A\dot{W}}{}^{C\dot{Y}}\phi_{C\dot{Y}B\dot{X}} + R\phi_{AB\dot{W}\dot{X}} - 3/16\, R^2\varepsilon_{AB}\varepsilon_{\dot{W}\dot{X}} = 0. \tag{5.13}$$

In dyad components this is

$$4\phi_{a\dot{w}}{}^{c\dot{y}}\phi_{c\dot{y}b\dot{x}} + R\phi_{ab\dot{w}\dot{x}} - 3/16\, R^2\varepsilon_{ab}\varepsilon_{\dot{w}\dot{x}} = 0. \tag{5.14}$$

By considering different values of the dyad components, Eq. (5.14) gives the following set of equations:

$$8\phi_{00}\phi_{11} - 8\phi_{01}\phi_{10} + R\phi_{00} = 0 \tag{5.15}$$

$$4\phi_{00}\phi_{12} - 4\phi_{10}\phi_{02} + R\phi_{01} = 0 \tag{5.16}$$

$$8\phi_{01}\phi_{12} - 8\phi_{11}\phi_{02} + R\phi_{02} = 0 \tag{5.17}$$

$$4\phi_{11}^2 - 4\phi_{01}\phi_{12} - 4\phi_{01}\phi_{21} + 4\phi_{00}\phi_{22} - 3/16\, R^2 + R\phi_{11} = 0 \tag{5.18}$$

$$4\phi_{01}\phi_{22} - 4\phi_{02}\phi_{21} + R\phi_{12} = 0 \tag{5.19}$$

$$8\phi_{11}\phi_{22} - 8\phi_{12}\phi_{21} + R\phi_{22} = 0. \tag{5.20}$$

Contracting the undotted and the dotted indices in Eq. (5.14), we get

$$R^2 = 32/3\, (\phi_{00}\phi_{22} - 2\phi_{01}\phi_{21} + \phi_{02}\phi_{20} - 2\phi_{10}\phi_{12} + 2\phi_{11}^2). \tag{5.21}$$

From the transformation laws, [Eq. (A.1) or (A.2)], for the dyad components, $\phi_{ab\dot{w}\dot{x}}$, it is easy to see that we can choose a

frame where $\phi_{02} = 0$. In this frame, Eq. (5.17) implies that either ϕ_{01} or ϕ_{12} is zero, or both of them are zero. If $\phi_{01} = 0$, then Eqs. (5.16), (5.19), (5.18) and (5.15) imply that $\phi_{12} = 0$ also. If $\phi_{12} = 0$, then Eqs. (5.19), (5.16), (5.18), and (5.15) imply that $\phi_{10} = 0$. Therefore, in this frame, we have $\phi_{02} = \phi_{12} = \phi_{10} = 0$. The Eqs. (5.15)-(5.21) reduce to

$$\phi_{00}(8\phi_{11} + R) = 0 \tag{5.22}$$

$$4\phi_{11}^2 + 4\phi_{00}\phi_{22} - 3/16\ R^2 + R\phi_{11} = 0 \tag{5.23}$$

$$\phi_{22}(8\phi_{11} + R) = 0 \tag{5.24}$$

$$R^2 = 32/3\ (\phi_{00}\phi_{22} + 2\phi_{11}^2). \tag{5.25}$$

Substituting Eq. (5.25) into (5.23), we obtain

$$2\phi_{00}\phi_{22} + R\phi_{11} = o \tag{5.26}$$

Since $\Phi_{AB\dot{W}\dot{X}}$ is not identically zero, Eqs. (5.26) and (5.23) imply that both ϕ_{00} and ϕ_{22} cannot be zero; and hence, from Eqs. (5.22) and (5.24), we have $R = -8\phi_{11}$. Eq. (5.26) then says that $4\phi_{11}^2 = \phi_{00}\phi_{22}$ and, therefore, ϕ_{00} and ϕ_{22} must have the same sign.

Let us define an Hermitian spinor $p_{A\dot{W}}$ as follows:

$$\pm p_0^2 = \phi_{00}$$

$$\pm p_2^2 = \phi_{22}$$

$$\pm\ 1/2\ p_0 p_2 = \phi_{11}$$

$$p_1 = 0.$$

The plus sign is used if and only if ϕ_{00} and ϕ_{22} are non-negative. ϕ_{00} and ϕ_{22} determine the magnitude of p_0 and p_2 respectively. ϕ_{11} determines the relative sign of p_0 and p_2. From the set of equations (4.14), we see that $S_{\alpha\beta}$ is in class I (either I-A <u>or</u> I-B). We can also note that

$$R = -8\phi_{11} = \pm 4p_0p_2 = \pm 2p_{A\dot{W}}p^{A\dot{W}},$$

in the frame where $\phi_{02} = 0$, and hence

$$R = \pm 2\mathbf{p}_\alpha \mathbf{p}^\alpha \qquad (5.27)$$

in any frame.

Writing Eq. (5.10) in spinor form, we get

$$(-2\phi_{ab\dot{w}\dot{x}} - 1/4\, R\varepsilon_{ab}\varepsilon_{\dot{w}\dot{x}})S^{a\dot{w}}S^{b\dot{x}} \leq 0. \qquad (5.28)$$

By taking $S^{1\dot{1}} = S^{2\dot{2}} = 1$ and $S^{1\dot{2}} = 0$ then, in a frame where $\phi_{02} = 0$, Eq. (5.28) reduces to

$$-2(\phi_{00} + 2\phi_{11} + \phi_{22}) \leq 1/2\; R. \qquad (5.29)$$

Using $R = -8\phi_{11}$ and the condition that ϕ_{00} and ϕ_{22} be the same sign, we get from Eq. (5.29) that ϕ_{00} and ϕ_{22} must be non-negative. Therefore, $S_{\alpha\beta}$ must be in class I-A and we must choose the negative sign in Eq. (5.27). By defining $t_\alpha = \sqrt{2}\mathbf{p}_\alpha$, we therefore see that Eqs. (5.9) and (5.10) are necessary and sufficient conditions for $R_{\alpha\beta}$ to satisfy Eq. (5.5).

An easy calculation shows that Eq. (5.12) is equivalent to Eq. (5.8) when R is not zero, and vanishes identically when R is

zero. We need not consider this lengthy expression since Eq. (5.8) is a geometrical condition; and, in addition, holds for all values of R.

We then have shown that Eq. (5.9) implies that $S_{\alpha\beta}$ is in class I (I-A <u>or</u> I-B), and the condition (5.10) selects the class I-A. Eq. (5.12) is equivalent to the compatability conditions, Eq. (5.8), when $R \neq 0$.

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APPENDIX I

1. Transformation Laws for the Dyad Components of $\Phi_{AB\dot{W}\dot{X}}$:

(i) Null Rotation about k^A; Eq. (1.13).

$$\phi'_{00} = \phi_{00}$$

$$\phi'_{01} = \phi_{00}\bar{c} + \phi_{01}$$

$$\phi'_{02} = \phi_{00}\bar{c}^2 + 2\phi_{01}\bar{c} + \phi_{02}$$

$$\phi'_{11} = \phi_{00}c\bar{c} + \phi_{01}c + \phi_{10}\bar{c} + \phi_{11}$$

$$\phi'_{12} = \phi_{00}c\bar{c}^2 + \phi_{10}\bar{c}^2 + 2\phi_{01}c\bar{c} + \phi_{02}c + 2\phi_{11}\bar{c} + \phi_{12}$$

$$\phi'_{22} = \phi_{00}c^2\bar{c}^2 + 2\phi_{01}c^2\bar{c} + 2\phi_{10}c\bar{c}^2 + \phi_{02}c^2 + \phi_{20}\bar{c}^2 + 4\phi_{11}c\bar{c} + 2\phi_{12}c + 2\phi_{21}\bar{c} + \phi_{22}. \tag{A.1}$$

(ii) Null Rotation About m^A; Eq. (1.14)

$$\phi'_{00} = \phi_{22}b^2\bar{b}^2 + 2\phi_{21}b^2\bar{b} + 2\phi_{12}\bar{b}^2b + \phi_{20}b^2 + \phi_{02}\bar{b}^2 + 4\phi_{11}b\bar{b} + 2\phi_{10}b + 2\phi_{01}\bar{b} + \phi_{00}$$

$$\phi'_{01} = \phi_{22}b^2\bar{b} + \phi_{21}b^2 + 2\phi_{12}b\bar{b} + 2\phi_{11}b + \phi_{02}\bar{b} + \phi_{01}$$

$$\phi'_{02} = \phi_{22}b^2 + 2\phi_{12}b + \phi_{02}$$

$$\phi'_{11} = \phi_{22}b\bar{b} + \phi_{21}b + \phi_{12}\bar{b} + \phi_{11}$$

$$\phi'_{12} = \phi_{22}b + \phi_{12}$$

$$\phi'_{22} = \phi_{22}. \tag{A.2}$$

(iii) $k'^A = ak^A$, $m'^A = 1/a\ m^A$.

$$\phi'_{00} = a^2\bar{a}^2\phi_{00}$$

$$\phi'_{01} = a^2\phi_{01}$$

$$\phi'_{02} = a^2\bar{a}^{-2}\phi_{02}$$

$$\phi'_{11} = \phi_{11}$$

$$\phi'_{12} = \bar{a}^{-2}\phi_{12}$$

$$\phi'_{22} = a^{-2}\bar{a}^{-2}\phi_{22}.$$

2. Proof of Theorem in Chapter IV:

By comparing coefficients in Eq. (4.4), we get the following equations:

$$
\begin{aligned}
2\phi_{00} &= T_{1\dot{1}}P_{1\dot{1}} + \bar{T}_{1\dot{1}}\bar{P}_{1\dot{1}} \\
4\phi_{01} &= T_{1\dot{1}}P_{1\dot{2}} + T_{1\dot{2}}P_{1\dot{1}} + \bar{T}_{1\dot{1}}\bar{P}_{1\dot{2}} + \bar{T}_{1\dot{2}}\bar{P}_{1\dot{1}} \\
2\phi_{02} &= T_{1\dot{2}}P_{1\dot{2}} + \bar{T}_{1\dot{2}}\bar{P}_{1\dot{2}} \\
8\phi_{11} &= T_{1\dot{1}}P_{2\dot{2}} + T_{1\dot{2}}P_{2\dot{1}} + T_{2\dot{1}}P_{1\dot{2}} + P_{1\dot{1}}T_{2\dot{2}} + \bar{T}_{1\dot{1}}\bar{P}_{2\dot{2}} + \bar{T}_{2\dot{1}}\bar{P}_{1\dot{2}} \\
&\quad + \bar{T}_{1\dot{2}}\bar{P}_{2\dot{1}} + \bar{P}_{1\dot{1}}\bar{T}_{2\dot{2}} \\
4\phi_{12} &= T_{1\dot{2}}P_{2\dot{2}} + T_{2\dot{2}}P_{1\dot{2}} + \bar{T}_{1\dot{2}}\bar{P}_{2\dot{2}} + \bar{T}_{2\dot{2}}\bar{P}_{1\dot{2}} \\
2\phi_{22} &= T_{2\dot{2}}P_{2\dot{2}} + \bar{T}_{2\dot{2}}\bar{P}_{2\dot{2}}
\end{aligned}
\tag{A.3}
$$

$(\phi_{10} = \overline{\phi_{01}}, \ \phi_{20} = \overline{\phi_{02}}, \ \phi_{21} = \overline{\phi_{12}})$.

We must show that we can always choose $T_{a\dot{w}}$ and $P_{a\dot{w}}$ to satisfy the above equations. To do this, let us make the following definitions:

$$
\begin{aligned}
T_{1\dot{1}} &= t_0 + is_0 & \qquad P_{1\dot{1}} &= p_0 + iq_0 \\
T_{1\dot{2}} &= t_1 + is_1 & \qquad P_{1\dot{2}} &= p_1 + iq_1 \\
T_{2\dot{1}} &= t_2 + is_2 & \qquad P_{2\dot{1}} &= p_2 + iq_2 \\
T_{2\dot{2}} &= t_3 + is_3 & \qquad P_{2\dot{2}} &= p_3 + iq_3
\end{aligned}
$$

$$\phi_{01} = a_1 + ib_1, \ \phi_{12} = a_2 + ib_2, \ \phi_{02} = a_3 + ib_3.$$

By choosing

$$q_0 = t_1 = t_2 = t_3 = s_0 = s_1 = s_2 = 0 \tag{A.4}$$

and

$$t_0 = s_3 = 1, \tag{A.5}$$

the set of equations (A.3) reduce to:

$$\begin{aligned}
\phi_{00} &= p_0 \\
4a_1 &= p_1 + p_2 \\
4b_2 &= p_1 - p_2 \\
4b_1 &= q_1 - q_2 \\
4a_2 &= -q_1 - q_2 \\
4\phi_{11} &= p_3 \\
\phi_{22} &= -q_3.
\end{aligned} \tag{A.6}$$

It is evident that we can solve this set of equations. Thus, with the $T_{a\dot{w}}$ and $P_{a\dot{w}}$ determined by Eqs. (A.4), (A.5), and the solutions of (A.6), we have satisfied equations (A.3) and hence have proven the theorem. It is obvious that this solution is not unique.

APPENDIX II

Canonical Forms for Classes II-A to II-F:

We shall reduce $T_{A\dot{W}}$ and $P_{A\dot{W}}$ to their simplest forms. In each case, the corresponding canonical form of $\Phi_{AB\dot{W}\dot{X}}$ can be obtained immediately from Eqs. (4.17).

Class II-A. $T^2 = 0$, $P^2 = 0$.

Choose the basis spinor m_A to correspond to the null direction P_α, and k_A to correspond to the null direction T_α. Then $(P_0, 0, 0)$ and $(0, 0, T_2)$ are the simplest forms of both $T_{A\dot{W}}$ and $P_{A\dot{W}}$.

Class II-B. $T^2 < 0$, $P^2 = 0$.

Choose m_A to correspond to the null direction P_α, and k_A to correspond to any other null direction orthogonal to T_α. In this dyad, the dyad components of P_α and T_α are $(P_0, 0, 0)$ and $(0, T_1, T_2)$, respectively. If, in addition, P_α and T_α are perpendicular then T_2 vanishes also.

Class II-C. $T^2 > 0$, $P^2 = 0$.

Again, by choosing m_A to correspond to the null direction P_α, the dyad components become $(P_0, 0, 0)$. Make a null rotation about m_A and choose the parameter "b" in Eq. (3.5b) so that $T_1' = 0$. For such a transformation, the form of $P_{A\dot{W}}$ is preserved. Hence, $(P_0, 0, 0)$ and $(T_0, 0, T_2)$ are the simplest forms for both $P_{A\dot{W}}$ and

$T_{A\dot{W}}$.

Class II-4. $T^2 < 0,\ P^2 > 0.$

If we choose a frame where $(0,\ T_1,\ 0)$ and $(P_0,\ P_1,\ P_2)$ are the dyad components of $T_{A\dot{W}}$ and $P_{A\dot{W}}$, we can make a null rotation about k^A and choose c so that

$$T_2' = T_1 c + \overline{T}_1 \overline{c} = 0. \tag{A.7}$$

We then have

$$T_0' = 0,\ T_1' = T_1,\ T_2' = 0,$$

and

$$P_0' \neq 0,\ P_1' = P_0 \overline{c} + P_1,\ P_2' \neq 0.$$

When will "c", in Eq. (A.7), make $P_1' = 0$ also? Let us define $T_1 = a + ib$, $P_1 = e + if$, and $c = x + iy$. Then Eq. (A.7) implies

$$2(ax + by) = 0. \tag{A.8}$$

For P_1' to be zero, we must have

$$c \equiv x + iy = -P_1/P_0. \tag{A.9}$$

From Eqs. (A.8) and (A.9), we then have that $P_1' = 0$ if and only if

$$2(ae + bf) = 0.$$

Moreover, in this frame, we see from Eq. (4.18) that

$$P_{A\dot{W}}T^{A\dot{W}} = -T_1'\overline{P}_1' - P_1'\overline{T}_1' = -2(ae + bf),$$

and therefore, $P_1' = 0$ if and only if P_α and t_α are orthogonal.

Class II-5. $T^2 < 0,\ P^2 < 0.$

From the discussion in Chapter III, we see that any point

on the circle Γ (Eq. (3.8)) can be used to find a null vector orthogonal to $T_{A\dot{W}}$. Similarly, any point on the circle π, satisfying the equation (3.8), with $T_{a\dot{w}}$ replaced by $P_{a\dot{w}}$, corresponds to a null direction perpendicular to $P_{A\dot{W}}$. ($P_1 \equiv c + id$). To reduce both $T_{A\dot{W}}$ and $P_{A\dot{W}}$ to their simplest forms, we shall consider the following possibilities:

(i) Γ and π intersect at two points.

(ii) They intersect at only one point.

(iii) They have no points in common.

These three cases will be distinguished by an invariant.

The square of the distance between the centers of Γ and π is:

$$D^2 = (a/T_0 - c/P_0)^2 + (d/P_0 - b/T_0)^2.$$

With a proper transformation, Eq. (3.8) and the corresponding equation for $P_{A\dot{W}}$ can be put in the form

$$x^2 + y^2 = r_1^2, \qquad \text{(A.10)}$$

and

$$(x - D)^2 + y^2 = r_2^2. \qquad \text{(A.11)}$$

Solving Eqs. (A.10) and (A.11), we get

$$x = r_1^2 + D^2 - r_2^2/2D \qquad \text{(A.12)}$$

and

$$y = \pm[4D^2 r_1^2 - (r_1^2 + D^2 - r_2^2)^2]^{1/2}/2D. \qquad \text{(A.13)}$$

If we define

$$R = 4D^2 r_1^2 - (r_1^2 + D^2 - r_2^2)^2,$$

then a straightforward calculation shows that

$$R = (P_{A\dot{W}}P^{A\dot{W}})(T_{B\dot{X}}T^{B\dot{X}}) - (P_{A\dot{W}}T^{A\dot{W}})^2.$$

If $R > 0$, then Eqs. (A.12) and (A.13) imply that Γ and π intersect at two points. We can then choose a dyad so that k^α <u>and</u> m^α are orthogonal to <u>both</u> t_α and P_α. In this frame, $T_{A\dot{W}}$ and $P_{A\dot{W}}$ have the forms $(0, T_1, 0)$ and $(0, P_1, 0)$ respectively.

If $R = 0$, then there is only one null direction perpendicular to <u>both</u> T_α and P_α. Choosing m^A to correspond to this null direction, and k^A to correspond to any other null direction orthogonal to T_α, we have $(0, T_1, 0)$ and $(P_0, P_1, 0)$ as the simplest forms of both $T_{A\dot{W}}$ and $P_{A\dot{W}}$.

With $R < 0$, we see that Γ and π have no points in common. We can choose a dyad in which $T_{A\dot{W}}$ and $P_{A\dot{W}}$ reduce to $(0, T_1, 0)$ and (P_0, P_1, P_2).

<u>Class II-6. $T^2 > 0$, $P^2 > 0$.</u>

The following argument will show that there exists a dyad in which both T_1 and P_1 vanish.

For a null rotation about k^A, we get

$$T_1' = T_1 + T_0\bar{c},\ T_0' \neq 0,\ T_2' \neq 0$$

and

$$P_1' = P_1 + P_0\bar{c},\ P_0' \neq 0,\ P_2' \neq 0.$$

Not yet specifying "c", make a null rotation about m^A. Then, we have

$$T_1'' = T_1' + T_2'b = T_1 + T_0\bar{c} + T_2'b, \quad T_0'' \neq 0, \quad T_2'' \neq 0, \qquad (A.14)$$

and

$$P_1'' = P_1' + P_2'b = P_1 + P_0\bar{c} + P_2'b, \quad P_0'' \neq 0, \quad P_2'' \neq 0. \qquad (A.15)$$

Multiplying Eq. (A.14) by P_2', Eq. (A.15) by T_2', and subtracting, we get

$$(T_0P_2' - P_0T_2')\bar{c} + T_1P_2' - P_1T_2' = 0. \qquad (A.16)$$

Using the definition of T_2' and P_2', Eq. (A.16) becomes

$$(\bar{P}_1T_0 - P_0\bar{T}_1)\bar{c}^2 + (T_0P_2 - P_0T_2 + T_1\bar{P}_1 - P_1\bar{T}_1)\bar{c} + T_1P_2 - P_1T_2 = 0. \qquad (A.17)$$

Not both the coefficients of $\bar{c}^2$ and $\bar{c}$ in Eq. (A.17) can be zero, since $T_{A\dot{W}}$ and $P_{A\dot{W}}$ are not proportional. Therefore, choosing c to satisfy Eq. (A.17), and

$$b = -T_1 - T_0\bar{c}/T_2',$$

we see that $T_1'' = P_1'' = 0$.